Solution 2, by Somasundaram Muralidharan.

Let $f(x) = 16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1$. Let us first check whether f has repeated roots. Such repeated roots, if any, will be roots of $\gcd(f(x), f'(x))$, where $f'(x)$ is the derivative of $f(x)$. In this case, it is easy to see that $\gcd(f(x), f'(x)) = 2x^2 - 2x + 1$ and hence the roots of this gcd, namely $\frac{1 \pm i}{2}$, are double roots of $f(x) = 0$. Thus we have found four of the roots of f. We now find the remaining two roots of f. We have

$$f(x) = (2x^2 - 2x + 1)^2(4x^2 + 2x - 1)$$

and hence the remaining roots are roots of $4x^2 + 2x - 1 = 0$. These are $\frac{-1 \pm \sqrt{5}}{4}$. So, the complex roots of f are

$$\frac{-1 + \sqrt{5}}{4}, \ \frac{-1 - \sqrt{5}}{4}, \ \frac{1 + i}{2}, \ \frac{1 + i}{2}, \frac{1 - i}{2}, \ \frac{1 - i}{2}.$$

4225. *Proposed by Leonard Giugiuc, Daniel Dan and Daniel Sitaru.*

Prove that in any triangle ABC we have:

$$3(\cos^2 A + \cos^2 B + \cos^2 C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3.$$

We received 13 correct solutions. We present the solution by Arkady Alt.

Since $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$ in any triangle ABC, the original inequality is successively equivalent to

$$\begin{aligned} &3(1 - 2\cos A \cos B \cos C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3 \\ \iff &\cos A \cos B + \cos A \cos C + \cos B \cos C \geq 6 \cos A \cos B \cos C. \end{aligned}$$

Setting $t = \sqrt[3]{\cos A \cos B \cos C}$ and using the AM-GM inequality, we obtain

$$\begin{aligned} 1 - 2t^3 &= 1 - 2 \cos A \cos B \cos C \\ &= \cos^2 A + \cos^2 B + \cos^2 C \\ &\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C} \\ &= 3t^2. \end{aligned}$$

Therefore $2t^3 + 3t^2 - 1 \leq 0$, implying successively that $(2t-1)(t+1)^2 \leq 0$ and then $t \leq \frac{1}{2}$. Hence, $2 \cdot \sqrt[3]{\cos A \cos B \cos C} \leq 1$, and again by the AM-GM inequality we have

$$\begin{aligned} \cos A \cos B + \cos A \cos C + \cos B \cos C &\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C} \\ &\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C} \cdot \sqrt[3]{\cos A \cos B \cos C} \\ &= 6 \cos A \cos B \cos C. \end{aligned}$$